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NEIGHBORHOODS AND COVERING VERTICES BY CYCLES

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In this work, we show the following results. First if $G=(V,E)$ is a 2-connected graph, and X is a set of vertices of G such that for every pair x,x' in X, $|N_G(x)\cup N_G(x')|\geq n/2+2$, and the minimum degree of the induced graph $<X>$ is at least 3, then X is covered by one cycle.

This result will be in fact generalised by considering tuples instead of pairs of vertices.

Let $\delta_1(X)$ be the minimum degree in the induced graph $<X>$. For any $t\geq 2$,

$\delta_t(X)=\min\{|N_G(u_1)\cup N_G(u_2)\ldots\cup N_G(u_t)|, u_i\neq u_j, u_1,\ldots,u_t\in X\}$.

If $\delta_1(X)\geq$ t, and $\delta_t(X)\geq |V|/p+t$, then X is covered by at most $(p-1)$ cycles of G. If furthermore $\delta_1(X)\geq 2t$, $(p-1)$ cycles are sufficient.

So we deduce the following:

Let p and t $(t\geq 2)$ be two integers.

Let G be a 2-connected graph of order n, of minimum degree at least t. If $\delta_1\geq$ t, and $\delta_t\geq n/p+t$, then V is covered by at most $(p-1)+\lceil (t-1)/k\rceil$ cycles, where k is the connectivity of G.

If furthermore $\delta_1\geq 2t$, $(p-1)$ cycles are sufficient.

In particular, if $\delta_1\geq 2t$ and $\delta_t\geq n/2+t$, then G is hamiltonian.

1. Introduction

In this work we consider finite and undirected simple graphs. If $G=(V,E)$ is a graph we denote $N_G(z)$ the neighborhood of a vertex z i.e. the set of vertices of G adjacent to z.

Let $N_G[a]=N_G(a)\cup\{a\}$, $N_G(a\cup b)=N_G(a)\cup N_G(b)$, and $\delta_2=\min\{|N_G(u)\cup N_G(v)|, u,v\in V$ and $u\neq v\}$.

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For any subset X of V, then $<X>$ is the subgraph of G induced by the set X, $\delta_1(X)$ is the minimum degree in the graph $<X>$. For any $t \geq 2$, $\delta_t(X) = \min\{|N_G(u_1) \cup N_G(u_2) \ldots \cup N_G(u_t)|, u_i \neq u_j, u_1, \ldots, u_t \in X\}$, and

$$\sigma_t(X) = \min\{\sum_{x \in S} \deg_G x | S \subseteq X \text{ is an independent set in } G \text{ and } |S| = t\}.$$

Note that $\sigma_t(X) \geq \delta_t(X)$. If $X = V$, we will denote $\delta_t(X)$ by simply δ_t.

A p-cycle cover of $X \subset V$ in the graph G is a family of p cycles of the graph G such that each element of X belongs to at least one of the cycles of this family. We say also that X is covered by p cycles.

The maximum length of a cycle in G is called circumference of G.

Let C be a cycle. A path $Q[a,b]$ of endpoints a and b is said to be strongly joined to the cycle C if Q has no vertex in common with C and there exist 2 vertex-disjoint paths $P[a,c]$ and $P[b,c']$ from $Q[a,b]$ to C internally disjoint from C. These two last paths are internally vertex disjoint from $Q[a,b]$. Then $Q[a,b]$ is called a C-path and the vertices c and c' are called joins of the C-path $Q[a,b]$.

We suppose that C has an orientation and a vertex labeling following this orientation. We denote by $[a,b]$ (resp. $[a,b[$) the segment of C consisting of the vertices x such that $a \leq x \leq b$ (resp. $a \leq x < b$). We denote by $]a,b[$ the segment $[a,b] \setminus \{a,b\}$. Two chords au and bu' of C are crossings if we meet succecively (a,u',u,b) or (a,b,u,u').

We recall the definition of a quasi-claw free graph H. For each pair of vertices a,b of H at distance 2, there exists at least a vertex $u \in N[a] \cap N[b]$ such that $N[u] \subset N[a] \cup N[b]$.

Several authors studied relations between the parameter δ_2 and hamiltonicity, or the circumference of the graph. It has been shown that

Theorem A. [3] *Let G be a 2-connected graph of order n. If $\delta_2 \geq n/2$, then G is hamiltonian for n sufficiently large.*

Jackson [4] has proved that a 3-connected graph with $\sigma_2 \geq (n+1)/2$ is hamiltonian.

Theorem B. [2] *If G is a 2-connected graph of order n, $n \geq 10$, and $\delta_2 \geq 2n/5+1$, then there exists a 2-cycle cover of G. Furthermore, one of the cycles can be chosen as a longest cycle of G.*

On the other hand, we have also the result [5]:

"If $\sigma_t(G) \geq n$, then G is covered by at most $(t-1)$ subgraphs, each of them is a cycle, an edge or a vertex". More exactly,

Theorem C. *[5] Let $k \geq 2$ be an integer. Let G be a graph on n vertices and let $X \subseteq V$. If $\sigma_t(X) \geq n$ or $\alpha(X) < t$ then X is covered with $t-1$ cycles, edges or vertices of G.*

2. Main results

First we obtain the following results for 2-connected graphs.

Theorem 1. *Let p $(p \geq 2)$ be any integer. Let G be a 2-connected graph of order n, and of minimum degree 3. If $\delta_2 \geq n/p+2$, then the vertices of G are covered by at most $p-1$ cycles.*

This theorem is sharp. Let us remark that the Petersen graph satisfies $\delta_2 = n/2$ but is not hamiltonian.

For $p = 2$ and 3, this theorem gives an improvement of Theorems B and C. In the particular case where G is a quasi-claw-free graph, this result recalls a result of [1]. In fact, with the hypotheses of the theorem, we have $\sigma_{2p}(G) \geq n$. In a set of $2p$ vertices of G, either there exist 2 adjacent vertices or 2 vertices at distance 2. So $\alpha(G^2) \leq 2p–1$. Then by [1], as G is 2 connected, V is covered by at most p cycles.

To establish Theorem 1, we prove the following basic result:

Proposition 1. *Let $G = (V, E)$ be a 2-connected graph of order n, and X a set of vertices G. Suppose that, for every pair x, x' in X, $|N_G(x) \cup N_G(x')| \geq n/2+2$ and the minimum degree of $< X >$ is at least 3. Then X is covered by one cycle of G.*

We generalize Theorem 1 as follows.

Theorem 2. *Let p and t $(p \geq 2, t \geq 3)$ be two integers. Let G be a 2-connected graph of order n, such that $\delta_t \geq n/p+t$.*

1) *If the minimum degree of G is at least t, then the vertices of G are covered by at most $(p-1)+\lceil (t-1)/k \rceil$ cycles, where k is the connectivity of G.*

2) *If, furthermore, the minimum degree of G is at least $2t$, then the vertices of G are covered by at most $(p-1)$ cycles.*

This theorem is sharp. Let k $(k \geq 2)$ be an integer. Let T be a set of k independent edges and let S be an independent set of $(k-1)$ other vertices. Let G_0 be the join of S and T. The graph G_0 is of minimum degree k. Then $\delta_k = (2k - 1) = k + (k-1)$, this value corresponds to the order of the neighborhood of k vertices of T. We have $n = 3k-1$ and $p = 3$. The theorem gives a 2-cycle cover of G_0, and this is in fact a minimum covering.

We prove Theorem 2 in the following form.

Proposition 2. *Let $p \geq 2$ and $t \geq 3$ be two integers. Let G be a 2-connected graph of order n, and X a set of vertices of G.*
1) If the minimum degree in the subgraph $<X>$ is at least t, and for each tuple in X, we have $|N_G(x_1) \cup N_G(x_2) \ldots \cup N_G(x_t)| \geq n/p + t$
then
X is covered by at most $(p-1) + \lceil (t-1)/k \rceil$ cycles, where k is the connectivity of G.
2) If furthermore, the minimum degree of $< X >$ is at least $2t$, then X is covered by at most $(p-1)$ cycles.

3. Proofs

We give now the proof of Proposition 1.

Proof of Proposition 1. The proof is by contradiction.

Let C be a cycle of G

i) containing the maximum number of vertices of X,

ii) and then, C is, among the cycles satisfying (i), of maximum length.

As C does not cover X, by the 2-connectivity of G, there exists a C path intersecting $X \setminus C$. We give an orientation to this cycle.

First Case. $X \setminus C$ is not an independent set.

So $X \setminus C$ contains at least 2 vertices. Let Q be a path such that

α) Q is strongly joined to C.

β) $|Q \cap X|$ is maximum,

Let x_1x_2 be an edge of $<X \setminus C>$. Then the path x_1x_2 is strongly joined to the cycle C. So, by maximality of Q, we have $|Q \cap X| \geq 2$. We may suppose $Q = Q[x, x']$ with x and x' vertices of X. Let c, c' be the joins of Q with C chosen so that there is no neighbor c'' of x or x' in $]c, c'[$ (γ).

By maximality of C, each of the segments $]c, c'[$ and $]c', c[$ intersect X.

Let a (respectively a') be the first vertex of X which follows c (respectively c') on the cycle.

And also, by maximality of C and by minimality (see γ) of $]c, c'[$, we have

$N_C(x \cup x') \cap N_C(a)^- \subset \{c, c'\}$ (1)

$N_C(x \cup x') \cap N_C(a')^- \subset \{c, c'\}$ (2).

Similarly, there is no neighbor contained in $G \backslash C$ common to some of the vertices x, x' and to some of a, a'. By hypothesis on X , we have

$$(*) \qquad |N_G(x \cup x')| \geq n/2 + 2$$

$$(**) \qquad |(N_{G \backslash C}(a \cup a'))| + |N_C(a \cup a')^-| \geq n/2 + 2.$$

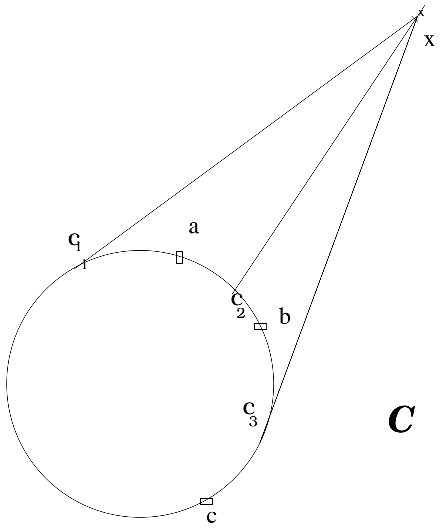

Fig. 1. The cycle C

By maximality of C, we observe that $N_C(x \cup x')$ does not contain a or a'; The set $A = (N_{G\setminus C}(a \cup a')\ \cup N_C(a \cup a')^-$ does not contain x or x'. So from $(*)$ and $(**)$, it follows that each of the sets A and $N_G(x \cup x')$ has at least $n/2$ vertices in $G \setminus \{a, a', x, x'\}$, they have an intersection of at least 4 vertices. We get a contradiction with the inequalities (1) and (2).

Case 1 is now excluded. Let $d(C) = \min\{d_C(c, c'),\ c \text{ and } c' \text{ pair of joins}\}$. If there are several choices, we choose C such that $d(C)$ is minimal.

Second Case. $X \setminus C$ is an independent set.

Let x be a vertex in $X \setminus C$. By hypothesis on X, the vertex x has at least three neighbors c_1, c_2, c_3 in X, then on the cycle C. We choose x such that the distance between two neighbors (say c_2, c_3) is minimum. Let a, b, c be the vertices of X which follow respectively c_1, c_2, c_3 on C.

Let $A = (N_{[c_1{}^{++}, b^-]}(b \cup c))^+$. $B = (N_{[c^+, c_1{}^+\ ]}(b \cup c))^+$, $C = (N_{[b^+, c]}(b))^-$, $D = (N_{[b, c^+]}(c))^+$.

Then,

α) by maximality of C, b has no neighbor in $]c_3, c[$, neither in $]c_1, a[$.

β) by definition of C there is no triple u^-, u, u^+ in $[b, c_3[$ such that u^- is neighbor of c, and u^+ is neighbor of b. Otherwise, either we get a longer cycle, or, a cycle C_1 with the same length as C and such that $d(C_1) < d(b, c_3) < d(C)$; we are in contradiction with the definition of C. We remark that $bc_3{}^+$ is not an edge. So, in the segment $[b, c^-]$, there is no common vertex to $(N(b))^-$ and $(N(c))^+$.

Thus $C \cap D$ is empty. It is obvious that $A \cap B$ is empty.

Consequently $|A \cup B| \geq |N_{[c^{++},b-]}(b \cup c) \cup N_{[c^{+},c_1{}^{+}]}(b \cup c)|$.

$|C \cup D| \geq |N_{[b^{+},c]}(b) \cup N_{[b,c-]}(c)|$

By maximality of C, we get $(A \cup B) \cap (C \cup D) \subset \{b \cup c\}$.

Using the hypothesis on the neighborhoods of X, we conclude that the set $M = A \cup B \cup C \cup D \cup N_{G \setminus C}(c \cup b)$ has at least $n/2$ vertices. (3)

On the other hand, by maximality of the cycle C, there is no crossing between 2 chords of C, of the form $au\ bu'$ (where $u' = u^-$ or u^+). Thus

$N_C(a) \cap (N_{[b^{+},c_1{}^{+}]}(b))^- = \emptyset$.

$N_C(a) \cap (N_{[c_1{}^{++},(b)-]}(b))^+ = \emptyset$.

Analogously the same holds when we replace b by c.

If we replace a by x in the last two sets, the only possible intersection is in $\{c_2, c_3\}$. Then

$N_C(a \cup x) \cap (A \cup B \cup C \cup D) \subset \{c_2, c_3\}$. (4)

Furthermore, $N_{G-C}(x \cup a) \cap N_{G-C}(c \cup b) = \emptyset$ (4′) .

By hypothesis on X, $|N_G(x \cup a)| \geq n/2 + 2$ vertices. The two sets M and $N_G(x \cup a)$ are subsets of $G \setminus \{x\}$. So, by (3), the intersection of these 2 sets must contain at least 3 vertices.

We get also a contradiction with (4) and (4’) •

Remark

1) The proposition is sharp. For example in the graph $K_{(p,p+1)}$, we cannot cover the stable set X of cardinality $p+1$ by one cycle; and $\delta_2 \geq (n-1)/2$.

2) We can replace the hypothesis of the 2-connectivity of G by the following one: “any two vertices of X are joined by two disjoint paths of G.”

Proof of Theorem 1.

We prove the Theorem in the following form.

“Let G be a graph of order n, and X a subset of G such that $\delta_1(X) \geq 3$. If any pair of vertices of X are joined by at least 2 vertex-disjoint paths of G, and if $\delta_2(X) \geq n/p + 2$, then X is covered by at most $p-1$ cycles of G.”

The proof is by induction on p. For $p = 2$, by Proposition 1, X is covered by one cycle.

From now $p \geq 3$. Let P be a path of G of maximum length with extremities in X. Let a and b be the extremities of P. We give an orientation to P from a to b.

Let $X(a)$ (resp. A(a)) be the set of vertices of X (resp. of G) which preced the neighbors of a on the path.

As the minimum degree of X is at least 3, then $|X(a)| \geq 2$.

Let a' be the first vertex of $X(a)$ different from a. There are 2 cases:

α) either $a' < z$ for every $z \in N_P(a)$, $z \neq a^+$.

Let $A(a') = \{z^+; z < a', \text{ and, } z \in N_P(a')\} \cup \{z^-, z > a' \text{ and } z \in N_P(a')\}$

β) or, there exists a neighbor z_1, of a, different from a^+ such that $z_1 < a'$. We choose z_1 as near as possible from a' and we set

$A_1(a') = \{z^-; z < z_1$, and, $z \in N_P(a')\} \cup \{z^+; z_1 < z < a'$, and $z \in N_P(a')\}$,

$A_2(a') = \{z^-; z > a'$, and, $z \in N_P(a')\}$ and $A(a') = A_1(a') \cup A_2(a')$.

Let us remark that there is at most one vertex, the vertex a' in the intersection of $A_1(a')$ and $A_2(a')$. By definition of a', the intersection of $A_1(a')$ and $A(a)$ has at most one element. We deduce that

$|A(a) \cup A(a')| \geq |N(a) \cup N(a')| - 1$.

In the path P, let $c' = p_i$ be the vertex, with i maximum, such that p_i is a neighbor of some vertex of $A(a) \cup A(a')$.

Let C be the cycle composed by the segment $[a, c']$ and one, two, or three segments depending if c' is respectively a neighbor of a, $A(a)$ or $A(a')$.

We remark that no vertex of $X \setminus C$ is neighbor of $A(a) \cup A(a')$ by maximality of P and construction of C.

Let $X' = X \setminus C$. Consider the set D of extremities on C of the paths from $G \setminus C$ to C. If D is of cardinality 1 or respectively 2 then in the case where $|C \setminus A(a) \cup A(a')| = |D|$, the cycle C is replaced by a vertex or respectively an edge.

In the general case, let us define the graph G' by removing the vertices of $A(a) \cup A(a')$, and adding edges uv whenever u and v are endpoints of a subpath of C consisting of removed vertices. This set is disjoint from $A(a) \cup A(a')$ in the graph G. The transformation of C gives a cycle, say C'. The order n' of G' is at most $n - n/p$, and $N_{G'}(X') = N_G(X)$. Then $\delta_2(G') \geq n'/(p-1) + 2$ and $\delta_1 < X' > \geq 3$.

One can verify that in the graph G' any 2 vertices of X' are joined by 2 vertex-disjoint paths.

By the induction hypothesis applied to X' in G', we cover X' by at most $(p-1)$ cycles, edges or vertices. •

Proof of Proposition 2.

Consider a path with extremities in X and such that

i) $|P \cap X|$ is maximum

ii) $|P \cap (V - X)|$ is minimum.

Let a and b be the extremities of P. We do the following construction. For each vertex $u \in X(a)$, consider $X_1(u)$ and $X_2(u)$, as in Theorem 1: $X_1(u)$ is the set of vertices which succeed to the neighbors of u in $P[a, u]$, $X_2(u)$ is the set of those who preceed the neighbors of u in $P[u, b]$.

If for some vertices v and v' of $X(a)$, the set $X_2(v') \cap X_1(v)$ is not empty, then we define $X(x)$ for every vertex in that intersection. We repeat this construction until no new vertex is given in the intersections of the form $X_2(v') \cap X_1(v)$. Let X'' be the set of vertices of X so obtained. For every v in X'', $X(v)$ is contained in X''.

Let $v_1 = a, \ldots, v_t$ be t consecutive vertices of X''.

Then $\sum_{i=1}^{t}(|A(v_i)| + |N_{G-P}(v_i))| \geq \sum_{i=1}^{t} |N(v_i)| - (t-1) \geq \delta_t(X) - (t-1)$, because the intersections of any pair of sets $A(v_i)$ is contained in $\{v_2, v_3, \ldots, v_t\}$.

Let x' be the last vertex of X''. Then there exists a cycle C which contains all the vertices of the segment $[a, x']$.

For $p = 2$,

1) Either $\delta_1 \geq t$, We construct the corresponding cycle and it remains $n/2$ vertices, which is less then δ_t. So it remains at most $t-1$ vertices in X. They are covered by at most $\lceil (t-1)/\kappa(G) \rceil$ cycles where $\kappa(G)$ is the connectivity of the graph G.

2) Or $\delta_1 \geq 2t$, from the vertex b we define similarly a set X'' and we take t consecutive vertices $w_1, \ldots, w_t = b$. As the minimum degree in $< X >$ is at least $2t$, we may suppose that $\{v_1, \ldots, v_t\} \cap \{w_1, \ldots, w_t\}$ is empty. We have

$$|\cup_{i=1}^{t} (A(v_i) \cup N_{G-P}(v_i))| \geq n/2 + 1 \text{ in } G - \{b, b^-\}.$$
$$|\cup_{i=1}^{t} (B(w_i) \cup N_{G-P}(w_i))| \geq n/2 + 1 \text{ in } G - \{a, a^-\}.$$

So there is an intersection between these two sets. The path P is contained in a cycle. We deduce that X is covered by one cycle.

For $p \geq 3$, consider the graph G' we obtain by removing $\cup_i (A(v_i))$ and adding edges uv whenever u and v are vertices of C bounding a segment of removed vertices. The order n' of G' is at most $n(p-1)/p$. Let X' be the set $X - \cup_i X(v_i)$. So $\delta_t(X') \geq n'/(p-1) + t$. We apply the induction hypothesis to G'.

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